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Discrete Mathematics 308 (2008) 427–438
DISCRETE MATHEMATICS
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Tight blocking sets in some maximum packings of λK_n ☆

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Received 31 October 2004; received in revised form 15 January 2005; accepted 27 November 2006
Available online 2 June 2007

Abstract

Let $(X, \mathscr{B})$ be a $(\lambda K_n, G)$-packing with edge-leave L and a blocking set T. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ be all connected components of L with at least two vertices (note that $s=0$ if $L=\emptyset$). The blocking set T is called tight if further $V(\Gamma_i) \cap T \neq \emptyset$ and $V(\Gamma_i) \cap (X \setminus T) \neq \emptyset$ for $1 \leqslant i \leqslant s$. In this paper we give a complete solution for the existence of a maximum $(\lambda K_n, G)$-packing admitting a blocking set (BS), or a tight blocking set (TBS) for any λ, and $G = K_3$, kite.

Keywords: G-design; Packing; Triple system; Kite system; Blocking set

1. Introduction

Let K_n be the complete graph with n vertices and λK_n denote the graph K_n with each of its edges replicated λ times. Given a family $\mathscr{G}$ of graphs each of which is simple and connected, a *λ-fold $\mathscr{G}$-design* (*or packing*) *of order n*, denoted by $(\lambda K_n, \mathscr{G})$-design (or packing) in short, is a pair $(X, \mathscr{B})$ where X is the vertex set of K_n, and $\mathscr{B}$ is a collection of subgraphs in λK_n, called *blocks*, such that each block is isomorphic to a graph in $\mathscr{G}$, and every edge $e \in E(\lambda K_n)$ belongs to exactly (or at most) λ blocks of $\mathscr{B}$. When $\mathscr{G}$ contains a single graph G, we speak of a *λ-fold G-design* (*or packing*) *of order n*, denoted by $(\lambda K_n, G)$-design (or packing). If $\lambda = 1$, we drop the term "1-fold".

Let G be a simple and connected graph and $(X, \mathscr{B})$ a $(\lambda K_n, G)$-packing. The *edge-leave* L of $(X, \mathscr{B})$ is the graph (X, L) where the edge $xy \in L$ with multiplicity m if xy appears in $\lambda - m$ blocks of $\mathscr{B}$. A $(\lambda K_n, G)$-packing $(X, \mathscr{B})$ is called *maximum* if there does not exist any $(\lambda K_n, G)$-packing $(X, \mathscr{B}')$ with $|\mathscr{B}| < |\mathscr{B}'|$. Clearly, any λ-fold G-design of order n is a maximum λ-fold G-packing of order n with edge-leave $L = \emptyset$.

A *blocking set* of a $(\lambda K_n, G)$-packing $(X, \mathscr{B})$ is a set $T \subseteq X$ such that $V(b) \cap T \neq \emptyset$ and $V(b) \cap (X \setminus T) \neq \emptyset$ for any block $b \in \mathscr{B}$, where $V(b)$ denote the vertex-set of block b. Various papers have dealt with the investigation of blocking sets in $(\lambda K_n, G)$-designs. For example, [2] and a long series of papers finally culminated in the settling of the existence of $(\lambda K_n, K_4)$-designs [4] and of (K_n, G)-designs for all connected graphs G with at most 5 edges [3,5,6] with blocking sets.

☆ Supported in part by NSFC Grant 10371002 for the first author, and by confin. M.U.R.S.T. "Strutture geometriche, combinatorie e loro applicazioni", P.R.A. and I.N.D.A.M.(G.N.S.A.G.A.) for the other authors.
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doi:10.1016/j.disc.2006.11.060

Table 1
Edge-leaves of maximum packings of λK_n with triangles

		n (mod 6)					
		0	1	2	3	4	5
$\lambda = 1$		1-Factor	$\emptyset$	1-Factor	$\emptyset$	Tripole	4-Cycle
$\lambda > 1$ (mod 6)	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	1	1-Factor	$\emptyset$	1-Factor	$\emptyset$	Tripole	D
	2	$\emptyset$	$\emptyset$	Double edge	$\emptyset$	$\emptyset$	Double edge
	3	1-Factor	$\emptyset$	G	$\emptyset$	Tripole	$\emptyset$
	4	$\emptyset$	$\emptyset$	D	$\emptyset$	$\emptyset$	D
	5	1-Factor	$\emptyset$	Tripole	$\emptyset$	Tripole	Double edge

A *tripole* is the graph on n vertices consisting of $(n-4)/2$ disjoint edges and a 3-star; G is a graph on n vertices with $(n+4)/2$ edges and odd vertex degrees; D is a graph with 4 edges and even vertex degrees.

Let $(X, \mathscr{B})$ be a $(\lambda K_n, G)$-packing with edge-leave L and a blocking set T. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ be all connected components of L with at least two vertices (note that $s = 0$ if $L = \emptyset$). The blocking set T is called *tight* if further $V(\Gamma_i) \cap T \neq \emptyset$ and $V(\Gamma_i) \cap (X \backslash T) \neq \emptyset$ for $1 \leqslant i \leqslant s$.

In what follows we will denote the copy of K_3 (triangle) with vertices a, b, c by abc, and the copy of $K_3 + e$ (kite) with vertices a, b, c, d and the dangling edge cd by (a, b, c)-d.

In this paper we give a complete solution for the existence of a maximum $(\lambda K_n, G)$-packing admitting a blocking set (BS), or a tight blocking set (TBS) for any λ, and $G = K_3$, kite.

2. Maximum packings of λK_n with triangles

Table 1 shows the edge-leaves of a $(\lambda K_n, K_3)$-packing (MPT(n, λ) in short). Note that when $L = \emptyset$ MPT(n, λ) is a λ-fold triple system (TS(n, λ) in short). In this section we study the existence of BS and TBS in MPT(n, λ) for each pair (n, λ). We begin with the following:

Lemma 2.1. *Let T be a BS in an* MPT(n, λ) *with edge-leave L. Then*

$$\lambda \frac{n(n-4)}{8} \leqslant |V(L)| = l.$$

Proof. Let $t = |T|$, then $\lambda[\binom{t}{2} + \binom{n-t}{2}] - q = \frac{1}{3}[\lambda\binom{n}{2} - l]$, with $0 \leqslant q \leqslant l$, hence $3\lambda t^2 - 3\lambda t n + \lambda n(n-1) + l - 3q = 0$. Therefore $9n^2\lambda^2 - 12\lambda(\lambda n(n-1) + l - 3q) \geqslant 0$ which gives the inequality $\lambda n(n-4)/8 \leqslant l$. $\square$

It is well known that when $L = \emptyset$, the only TS(n, λ) that admit BS are the TS$(3, \lambda)$ for all λ and the TS$(4, \lambda)$ for all even λ. When $L \neq \emptyset$, as a consequence of Lemma 2.1, if T is a BS in an MPT(n, λ), then:

$n = 4$	$\lambda \equiv 1 \pmod 2$	L is a 3-star
$n = 5$	$\lambda = 1$	L is a 4-cycle
	$\lambda = 2$	L is a double edge
	$\lambda = 4$	L is a graph of type D
$n = 6$	$\lambda = 1$	L is a 1-factor
$n = 8$	$\lambda = 1$	L is a 1-factor

Case $n = 4$: It is easy to produce MPT$(4, \lambda)$ for all odd λ with TBS of size 2. One example when $\lambda = 5$ is:

Example 2.2. An MPT$(4, 5)$ with TBS: the triangles are $3\{123\} \cup 2\{124, 134, 234\}$; the edge-leave is the 3-star $\{\{4, 1\}, \{4, 2\}, \{4, 3\}\}$; a TBS is $T = \{1, 2\}$.

Case $n = 5$:

Example 2.3. An MPT(5, 1) with TBS: the triangles are $\{123, 145\}$; the edge-leave is the 4-cycle $(2, 4, 3, 5)$; a TBS is $T = \{2, 4\}$.

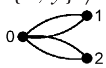
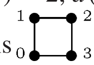
For the remaining cases when $v = 5$, there is the following:

Lemma 2.4. *Any* MPT$(5, \lambda)$, $\lambda = 2, 4$ *does not admit a TBS.*

Proof. Let $(Z_5, \mathscr{P})$ be an MPT$(5, \lambda)$, $\lambda = 2, 4$, and denote by $d(x)$ and $d(x, y)$ the number of triangles of $\mathscr{P}$ containing x and the 2-subset $\{x, y\}$, respectively. Let T be a TBS of $(Z_5, \mathscr{P})$; since $d(x) \leqslant \lambda(n-1)/2 < |\mathscr{P}|$, for each $x \in Z_5$, and being $Z_5 \backslash T$ also a TBS, we can suppose without loss of generality that $|T| = 2$.

Assume $\lambda = 2$. In this case the only admissible edge-leave, up to isomorphism, is the double edge $2\{0, 1\}$ and we have: $|\mathscr{P}| = 6$; $d(0) = d(1) = 3$ and $d(x) = 4$ for the remaining $x \in Z_5 \backslash \{0, 1\}$; $d(x, y) = 2$ when $\{x, y\} \neq \{0, 1\}$. Since T is a TBS, without loss of generality, we can suppose $T = \{0, 2\}$. It is easy to see that $143 \in \mathscr{P}$ and this is a contradiction.

Assume $\lambda = 4$. In this case $|\mathscr{P}| = 12$ and there are four admissible edge-leaves, up to isomorphism:

1. The edge-leave is  (isomorphic to $4K_2$); $d(0) = d(1) = 6$ and $d(x) = 8$ for the remaining $x \in Z_5 \backslash T$; $d(x, y) = 4$ when $\{x, y\} \neq \{0, 1\}$.
2. The edge-leave is  (isomorphic to $2K_{1,2}$); $d(0) = 6$; $d(1) = d(2) = 7$; $d(3) = d(4) = 8$; $d(0, 1) = d(0, 2) = 2$, $d(x, y) = 4$ for the remaining 2-subsets of Z_5.
3. The edge-leave is   (isomorphic to $2\{2K_2\}$); $d(0) = d(1) = d(2) = d(3) = 7$; $d(4) = 8$; $d(0, 1) = d(2, 3) = 2$, $d(x, y) = 4$ for the remaining 2-subsets of Z_5.
4. The edge-leave is  (a 4-cycle); $d(0) = d(1) = d(2) = d(3) = 7$; $d(4) = 8$; $d(0, 1) = d(1, 2) = d(2, 3) = d(3, 0) = 3$, $d(x, y) = 4$ for the remaining 2-subsets of Z_5.

In all four cases, by arguments similar to $\lambda = 2$ it is easy to show that the existence of a TBS gives a contradiction. □

Remark 2.5. We can deduce in a similar manner that any MPT(5, 4) with edge-leave a 4-cycle does not admit BSs.

Now we will produce some examples of MPT$(5, \lambda)$, $\lambda = 2, 4$, with BSs.

Example 2.6. An MPT(5, 2) with BS and edge-leave $2\{0, 1\}$: the triangles are $\{023, 024, 034, 123, 124, 134\}$; a BS is $T = \{0, 1\}$.

Example 2.7. An MPT(5, 4) with BS and edge-leave $4\{0, 1\}$: take two copies of Example 2.6.

Example 2.8. An MPT(5, 4) with BS and edge-leave $2\{\{0, 1\}, \{0, 2\}\}$: the triangles are $\{013, 014, 023, 024, 134, 234\} \cup 2\{034, 123, 124\}$; a BS is $T = \{3, 4\}$.

Example 2.9. An MPT(5, 4) with BS and edge-leave $2\{\{0, 1\}, \{2, 3\}\}$: the triangles are $\{012, 013, 023, 123\} \cup 2\{024, 034, 124, 134\}$; a BS is $T = \{0, 1\}$.

Cases $n = 6$ *and* 8: Let $(X, \mathscr{P})$ be an MPT$(n, 1)$, $n \equiv 0, 2 \pmod 6$. If we take $\mathscr{P}$ plus the triangles obtained joining a new point, say ∞, to each edge of the edge-leave L of $(X, \mathscr{P})$ (L is a 1-factor), then we get an STS$(n + 1, 1)$. Clearly, the existence of a TBS for MPT$(n, 1)$ leads to the existence of a BS for STS$(n + 1, 1)$ and so we have the following:

Lemma 2.10. *Any* MPT(6, 1) *and* MPT(8, 1) *do not admit a TBS.*

Table 2
Edge-leaves of maximum packings of λK_n with kites

		$n \geqslant 5 \pmod 8$							
		0	1	2	3	4	5	6	7
$\lambda \pmod 4$	1	$\emptyset$	$\emptyset$	D_1	D_3	D_2	D_2	D_3	D_1
	2	$\emptyset$	$\emptyset$	D_2	D_2	$\emptyset$	$\emptyset$	D_2	D_2
	3	$\emptyset$	$\emptyset$	D_3	D_1	D_2	D_2	D_1	D_3
	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

D_i, $i = 1, 2, 3$, is any graph with i edges (obviously, without parallel edges when $\lambda = 1$).

The following two examples are MPT(6, 1) and MPT(8, 1) with BS.

Example 2.11. An MPT(6, 1) with BS: the triangles are {123, 145, 246, 356}; a BS is $T = \{1, 6\}$.

Example 2.12. An MPT(8, 1) with BS: the triangles are {123, 147, 168, 258, 267, 348, 357, 456}; a BS is $T = \{1, 2, 4, 5\}$.

Now we are in position to present our main result of this section as follows.

Theorem 2.13. *There exists an* MPT(n, λ) *with* TBS *if and only if* $n = 4$ *and* λ *is odd or* $n = 5$ *and* $\lambda = 1$. *There exists an* MPT(n, λ) *with* BS *but without* TBS *if and only if* n *and* λ *are as in the following table*:

$n = 5$	$\lambda = 2$
	$\lambda = 4$ and L is isomorphic to $4K_2$
	$\lambda = 4$ and L is isomorphic to $2K_{1,2}$
	$\lambda = 4$ and L is isomorphic to $2\{K_2\}$
$n = 6$	$\lambda = 1$
$n = 8$	$\lambda = 1$

3. Maximum packings of λK_n with kites

When $L = \emptyset$ MPK(n, λ) is a *kite system* (KS(n, λ) in short). Table 2 shows all admissible edge-leaves for an MPK(n, λ), with the exception of the unique MPK(4, 1) whose edge-leave is the path $P_3(a_1, a_2, a_3)$. In this section we will give a complete solution of the problem of constructing for each pair (n, λ) a maximum packing of λK_n with kites (MPK(n, λ) in short) which admits TBS. More precisely, denote $N(n, \lambda)$ the set of all integers m such that there exists an MPK(n, λ) for *every* admissible edge-leave with a TBS of size m. We will prove that for every pair (n, λ) the set $N(n, \lambda)$ is nonempty.

3.1. Working lemmas

A *group-divisible design* (or GDD) with index λ is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the following properties:

(1) $\mathcal{G}$ is a partition of X into subsets called *groups*.
(2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point.
(3) Every pair of points from distinct groups occurs in exactly λ blocks.

The group-type of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. We also usually use an "exponential" notation to describe group-types: the group-type $1^i 2^j 3^k \ldots$ denotes i occurrences of 1, j occurrences of 2, etc. We call the GDD $(X, \mathcal{G}, \mathcal{A})$ a (K, λ)-GDD if $|A| \in K$ for every $A \in \mathcal{A}$. A $(\{k\}, \lambda)$-GDD is briefly written as (k, λ)-GDD.

We quote a known result for later use.

Lemma 3.1 (*see Colbourn and Rosa [1]*). *There exists a* $(3, 1)$-GDD *of type* 2^k *for every* $k \equiv 0, 1 \pmod 3$ *and a* $(3, 1)$-GDD *of type* $2^{k-2}4^1$ *for every* $k \equiv 2 \pmod 3 \geqslant 5$.

Lemma 3.2. *There exists a kite system of* $K_{4,4,4}$ (*i.e. a decomposition of* $K_{4,4,4}$ *into kites*) *with a BS of size* 6.

Proof. Let $(\{x, y, z\} \times Z_4, \mathscr{K})$ be the kite system of $K_{4,4,4}$ with the base $(K_3 + e)$-blocks in $(-, Z_4)$ as follows:

$$(z_0, x_1, y_0)\text{-}x_3, \quad (x_0, z_2, y_0)\text{-}z_1, \quad (z_1, y_2, x_0)\text{-}z_0.$$

The set $\{x, y, z\} \times \{0, 1\}$ is a BS of $(\{x, y, z\} \times Z_4, \mathscr{K})$. $\square$

Construction 3.3 (*Filling subdesigns*). *Let r be a nonnegative integer. Suppose that there exists a* $(3, \lambda)$-*GDD of group-type* $\{m_1, m_2, \dots, m_t\}$, $\sum_{i=1}^{t} m_i = 2k$. *If there is a kite system of* $\lambda(K_{4m_i+r} \backslash K_r)$, *for each* $2 \leqslant i \leqslant t$, *with a BS of size* $2m_i$ *with no vertices in the hole, then there exists a kite system of* $\lambda(K_{8k+r} \backslash K_{4m_1+r})$ *with a BS. If further* $2m_1 \in N(4m_1 + r, \lambda)$, *then* $4k \in N(8k + r, \lambda)$.

Proof. Suppose that $(X, \mathscr{G}, \mathscr{A})$ is a $(3, \lambda)$-GDD, where $\mathscr{G} = \{g_1, g_2, \dots, g_t\}$, $|g_i| = m_i$, and $\sum_{i=1}^{t} m_i = 2k$. Set $X = R \cup (X \times Z_4)$, where R is a set of size r, and define a collection $\mathscr{K}$ of kites as follows:

(1) For every $i = 2, 3, \dots, t$, let $(R \cup (g_i \times Z_4), \mathscr{K}_i)$ be a kite system of $\lambda(K_{4m_i+r} \backslash K_r)$ admitting $g_i \times \{0, 1\}$ as BS; put $\mathscr{K}_i \subseteq \mathscr{K}$.
(2) For every $t = \{x, y, z\} \in \mathscr{A}$, let$(\{x, y, z\} \times Z_4, \mathscr{K}_t)$ be the kite system of $K_{4,4,4}$ of Lemma 3.2; put $\lambda\mathscr{K}_t \subseteq \mathscr{K}$.

It easy to check that $(X, \mathscr{K})$ is a kite system of $\lambda(K_{8k+r} \backslash K_{4m_1+r})$ admitting $X \times \{0, 1\}$ as BS.

For *every* admissible edge-leave L, by the definition of $N(4m_1 + r, \lambda)$, we have an MPK$(4m_1 + r, \lambda)$ with edge-leave L and with $g_1 \times \{0, 1\}$ as TBS. Filling the hole of $(X, \mathscr{K})$ with the above MPK$(4m_1 + r, \lambda)$ gives an MPK$(8k + r, \lambda)$ with edge-leave L and with $X \times \{0, 1\}$ as TBS. $\square$

3.2. $\lambda = 1$

Here we will give examples of MPK$(n, 1)$ with *every* admissible edge-leave for small values of n admitting a TBS. In some particular cases we give also a BS that is not TBS (since it will be useful in the subsequent sections). Furthermore, we construct kite system of $K_{8+r} \backslash K_r$ for $r = 2, 3, 4, 5, 6, 7$ admitting a BS of size 4 with no vertices in the hole.

Let $K_s(a_1, a_2, \dots, a_s)$ denote the complete graph on vertex set $\{a_1, a_2, \dots, a_s\}$; $P_s(a_1, a_2, \dots, a_s)$ denote the path with $s - 1$ edges $a_i a_{i+1}$ for $1 \leqslant i \leqslant s - 1$; and $K_{1,s}(a_0; a_1, a_2, \dots, a_s)$ denote the graph with s edges $a_0 a_i$ for $1 \leqslant i \leqslant s$.

Lemma 3.4. $N(n, 1) \neq \emptyset$ *for every* $n = 8k + r \geqslant 4$, $k = 0, 1, 2$ *and* $0 \leqslant r \leqslant 7$. *In particular*, $4k \in N(8k + r, 1)$ *for* $k = 1, 2$ *and* $0 \leqslant r \leqslant 7$.

Proof. Let $(X, \mathscr{P})$ be an MPK$(n, 1)$ with edge-leave L as follows:

$n = 4$:
$X = \{0, 1, 2, 3\}$; $\mathscr{P} = \{(1, 2, 0)\text{-}3\}$; $L = \{P_3(2, 3, 1)\}$.
$T = \{0, 1\}$ is a TBS.
$n = 5$:
1. $X = \{0, 1, 2, 3, 4\}$; $\mathscr{P} = \{(1, 2, 0)\text{-}3, (2, 3, 4)\text{-}0\}$; $L = \{P_3(4, 1, 3)\}$.
2. Replace $(2, 3, 4)\text{-}0$ and $P_3(4, 1, 3)$ with $(2, 3, 4)\text{-}1$ and $L = \{K_2(4, 0), K_2(1, 3)\}$.
In both of the cases $T = \{0, 3\}$ is a TBS.
$n = 6$:
1. $X = \{0, 1, 2, 3, 4, 5\}$; $\mathscr{P} = \{(2, 5, 1)\text{-}0, (3, 0, 2)\text{-}4, (1, 3, 4)\text{-}0\}$; $L = \{K_{1,3}(5; 3, 4, 0)\}$.
2. $X = \{0, 1, 2, 3, 4, 5\}$; $\mathscr{P} = \{(0, 4, 1)\text{-}3, (4, 2, 3)\text{-}5, (1, 2, 5)\text{-}0\}$; $L = \{K_2(4, 5), P_3(2, 0, 3)\}$.

3. $X=\{0,1,2,3,4,5\}$; $\mathscr{P}=\{(2,4,3)\text{-}5, (1,4,0)\text{-}3, (1,2,5)\text{-}0\}$; $L=\{K_2(1,3), K_2(2,0), K_2(4,5)\}$.
4. $X=\{0,1,2,3,4,5\}$; $\mathscr{P}=\{(1,2,3)\text{-}4, (2,0,4)\text{-}1, (1,0,5)\text{-}2\}$; $L=\{P_4(4,5,3,0)\}$.
5. $X=\{0,1,2,3,4,5\}$; $\mathscr{P}=\{(2,5,1)\text{-}0, (3,0,2)\text{-}4, (1,4,3)\text{-}5\}$; $L=\{K_3(4,5,0)\}$.

In every of the previous cases $T=\{1,2,4\}$ is a TBS.

$n=7$:

$X=\{0,1,\ldots,6\}$; $\mathscr{P}=\{(2,3,5)\text{-}4, (0,5,1)\text{-}3, (3,4,6)\text{-}0, (1,2,6)\text{-}5, (2,4,0)\text{-}3\}$; $L=\{K_2(1,4)\}$.

$T=\{4,5,6\}$ is a TBS. $T'=\{2,5,6\}$ is a BS (not TBS).

$n=8$:

$X=Z_7\cup\{\infty\}$; $\mathscr{P}=\{(1+i,3+i,i)\text{-}\infty : i\in Z_7\}$; $L=\emptyset$.

$T=\{\infty,0,2,4\}$ is a TBS.

$n=9$:

$X=Z_9$; $\mathscr{P}=\{(1+i,3+i,i)\text{-}(4+i) : i\in Z_9\}$; $L=\emptyset$.

$T=\{0,2,4,6\}$ is a TBS.

$n=10$:

$X=Z_8\cup\{\infty_1,\infty_2\}$; $\mathscr{P}=\{(1+i,3+i,i)\text{-}\infty_1 : i\in Z_8\}\cup\{(4,0,\infty_2)\text{-}3, (5,1,\infty_2)\text{-}7, (6,2,\infty_2)\text{-}\infty_1\}$; $L=\{K_2(3,7)\}$.

$T=\{\infty_1,\infty_2,0,7\}$ is a TBS. $T'=\{\infty_1,\infty_2,0,6\}$ is a BS (not TBS).

$n=11$:

1. $X=Z_8\cup\{\infty_1,\infty_2,\infty_3\}$, $\mathscr{P}=\{(1+i,3+i,i)\text{-}\infty_1 : i\in Z_8\}\cup\{(4+i,\infty_2,i)\text{-}\infty_3 : 0\leqslant i\leqslant 3\}\cup\{(\infty_1,\infty_2,\infty_3)\text{-}4\}$, $L=\{K_{1,3}(\infty_3;5,6,7)\}$.
2. In 1 replace $(6,0,5)\text{-}\infty_1$ and $K_{1,3}(\infty_3;5,6,7)$ with $(6,0,5)\text{-}\infty_3$ and $L=\{K_2(\infty_1,5), P_3(6,\infty_3,7)\}$.
3. In 1 replace $(0,2,7)\text{-}\infty_1$, $(6,\infty_2,2)\text{-}\infty_3$, and $K_{1,3}(\infty_3;5,6,7)$ with $(0,2,7)\text{-}\infty_3$, $(2,\infty_3,6)\text{-}\infty_2$, and $L=\{K_2(\infty_1,7), K_2(\infty_2,2), K_2(\infty_3,5)\}$.
4. In 1 replace $(7,1,6)\text{-}\infty_1$ and $K_{1,3}(\infty_3;5,6,7)$ with $(1,6,7)\text{-}\infty_3$ and $\{P_4(5,\infty_3,6,\infty_1)\}$.
5. In 1 replace $(1,3,0)\text{-}\infty_1$, $(6,0,5)\text{-}\infty_1$, and $K_{1,3}(\infty_3;5,6,7)$ with $(5,6,\infty_3)\text{-}7$, $(1,3,0)\text{-}6$, and $L=\{K_3(5,0,\infty_1)\}$.

In every of the previous cases $T=\{6,\infty_1,\infty_2,\infty_3\}$ is a TBS.

$n=12$:

1. $X=Z_{11}\cup\{\infty\}$; $\mathscr{P}=\{(2+i,5+i,i)\text{-}(4+i) : i\in Z_{11}\}\cup\{(\infty,2+2i,3+2i)\text{-}(4+2i) : 0\leqslant i\leqslant 4\}$; $L=\{P_3(\infty,1,2)\}$.
2. Replace $(\infty,6,7)\text{-}8$ and $P_3(\infty,1,2)$ with $(6,7,\infty)\text{-}1$ and $L=\{K_2(1,2), K_2(7,8)\}$.

In both of the cases $T=\{2,6,7,10\}$ is a TBS.

$n=13$:

1. $X=Z_{10}\cup\{\infty_1,\infty_2,\infty_3\}$; $\mathscr{P}=\{(1+i,4+i,i)\text{-}\infty_1 : i\in Z_{10}\}\cup\{(\infty_2,i,2+i)\text{-}(4+i), (4+i,6+i,\infty_2)\text{-}(8+i) : i=0,1\}\cup\{(\infty_3,1+i,6+i)\text{-}(8+i) : i=0,1,2\}\cup\{(0,5,\infty_3)\text{-}9, (\infty_1,\infty_2,\infty_3)\text{-}4\}$; $L=\{P_3(4,9,1)\}$.
2. Replace $(1,4,0)\text{-}\infty_1$ and $P_3(4,9,1)$ with $(0,1,4)\text{-}9$ and $L=\{K_2(0,\infty_1), K_2(1,9)\}$.

In both of the cases $T=\{1,\infty_1,\infty_2,\infty_3\}$ is a TBS.

$n=14$:

1. $X=Z_{10}\cup\{\infty_1,\infty_2,\infty_3,\infty_4\}$, $\mathscr{P}=\{(1+i,3+i,i)\text{-}\infty_1 : i\in Z_{10}\}\cup\{(\infty_2,5+i,i)\text{-}(4+i) : 0\leqslant i\leqslant 4\}\cup\{(\infty_3,6+i,i)\text{-}\infty_4 : 0\leqslant i\leqslant 2\}\cup\{(5,9,\infty_3)\text{-}\infty_1, (\infty_4,9,3)\text{-}\infty_3, (\infty_1,\infty_2,\infty_4)\text{-}6, (\infty_4,4,\infty_3)\text{-}\infty_2\}$; $L=\{K_{1,3}(\infty_4;5,7,8)\}$.
2. In 1 replace $(6,8,5)\text{-}\infty_1$ and $K_{1,3}(\infty_4;5,7,8)$ with $(6,8,5)\text{-}\infty_4$ and $L=\{K_2(\infty_1,5), P_3(7,\infty_4,8)\}$.
3. In 1 replace $(6,8,5)\text{-}\infty_1$, $(\infty_3,7,1)\text{-}\infty_4$, and $K_{1,3}(\infty_4;5,7,8)$ with $(6,8,5)\text{-}\infty_4$, $(\infty_4,7,1)\text{-}\infty_3$, and $L=\{K_2(\infty_1,5), K_2(\infty_3,7), K_2(\infty_4,8)\}$.
4. In 1 replace $(6,8,5)\text{-}\infty_1$ and $K_{1,3}(\infty_4;5,7,8)$ with $(5,6,8)\text{-}\infty_4$ and $\{P_4(\infty_1,5,\infty_4,7)\}$.
5. In 1 replace $(6,8,5)\text{-}\infty_1$, $(5,7,4)\text{-}\infty_1$, and $K_{1,3}(\infty_4;5,7,8)$ with $(5,6,8)\text{-}\infty_4$, $(\infty_1,5,4)\text{-}7$, and $L=\{K_3(5,7,\infty_4)\}$.

In every of the previous cases $T=\{\infty_1,\infty_2,\infty_4,7\}$ is a TBS.

$n=15$:

$X=Z_{11}\cup\{\infty_1,\infty_2,\infty_3,\infty_4\}$; $\mathscr{P}=\{(3+i,5+i,i)\text{-}\infty_2 : i\in Z_{11}\}\cup\{(\infty_1,2i,1+2i)\text{-}(2+2i) : 0\leqslant i\leqslant 4\}\cup\{(\infty_3,5+i,9+i)\text{-}(2+i) : i=0,1,2\}\cup\{(1+i,5+i,\infty_4)\text{-}\infty_{1+i} : i=0,1,2\}\cup\{(\infty_4,0,10)\text{-}\infty_1, (\infty_4,8,4)\text{-}\infty_3, (8,1,\infty_3)\text{-}2, (\infty_1,\infty_2,\infty_3)\text{-}3\}$; $L=\{K_2(\infty_4,9)\}$.

$T=\{\infty_4,\infty_1,\infty_2,\infty_3\}$ is a TBS. $T'=\{0,\infty_1,\infty_2,\infty_3\}$ is a BS (not TBS).

$n=16$:

$X=Z_{15}\cup\{\infty\}$; $\mathscr{P}=\{(3+i,4+i,i)\text{-}\infty, (5+i,7+i,i)\text{-}(6+i) : i\in Z_{15}\}$; $L=\emptyset$.

$T=\{0,2,4,\ldots,14\}$ is a TBS.

$n=17$:

$X=Z_{17}$; $\mathscr{P}=\{(1+i,4+i,i)\text{-}(6+i),(5+i,7+i,i)\text{-}(8+i):i\in Z_{17}\}$; $L=\emptyset$.

$T=\{0,2,4,\dots,14\}$ is a TBS.

$n=18$:

$X=Z_{15}\cup\{\infty_1,\infty_2,\infty_3\}$; $\mathscr{P}=\{(3+i,10+i,i)\text{-}\infty_1,(6+i,2+i,i)\text{-}\infty_2:i\in Z_{15}\}\cup\{(2i,\infty_3,1+2i)\text{-}(2+2i):0\leqslant i\leqslant 6\}\cup\{(\infty_1,\infty_2,\infty_3)\text{-}14\}$; $L=\{K_2(0,14)\}$.

$T=\{\infty_1,\infty_2,\infty_3,0,1,3,5,12\}$ is a TBS. $T'=\{\infty_1,\infty_2,\infty_3,1,3,5,6,12\}$ is a BS (not TBS).

$n=19$:

0. $X=Z_{15}\cup\{\infty_1,\infty_2,\infty_3,\infty_4\}$, $\mathscr{P}=\{(1+i,4+i,i)\text{-}\infty_1,(2+i,8+i,i)\text{-}\infty_2:i\in Z_{15}\}\cup\{(\infty_3,5+i,i)\text{-}\infty_4,(\infty_4,5+i,10+i)\text{-}\infty_3:0\leqslant i\leqslant 4\}$; $L'=\{K_2(i,10+i):0\leqslant i\leqslant 4\}\cup\{K_2(\infty_i,\infty_j):\{i,j\}\subseteq\{1,2,3,4\},i\neq j\}$.

1. In 0 replace $(1,4,0)\text{-}\infty_1$, $(11,14,10)\text{-}\infty_1$, $(3,6,2)\text{-}\infty_1$, and L' with $(0,4,1)\text{-}11$, $(10,11,14)\text{-}4$, $(2,6,3)\text{-}13$, $(0,10,\infty_1)\text{-}2$, $(\infty_2,\infty_4,\infty_1)\text{-}\infty_3$, and $L=\{K_2(2,12),P_3(\infty_2,\infty_3,\infty_4)\}$.

2. In 1 replace $(4,10,2)\text{-}\infty_2$, $(\infty_2,\infty_4,\infty_1)\text{-}\infty_3$, and $\{K_2(2,12),P_3(\infty_2,\infty_3,\infty_4)\}$ with $(4,10,2)\text{-}12$, $(\infty_1,\infty_4,\infty_2)\text{-}2$, and $L=\{K_{1,3}(\infty_3;\infty_1,\infty_2,\infty_4)\}$.

3. In 0 replace $(3,6,2)\text{-}\infty_1$, $(\infty_3,7,2)\text{-}\infty_4$, $(1,4,0)\text{-}\infty_1$, $(11,14,10)\text{-}\infty_1$, and L' with $(2,6,3)\text{-}13$, $(0,10,\infty_1)\text{-}\infty_2$, $(\infty_3,7,2)\text{-}12$, $(\infty_4,2,\infty_1)\text{-}\infty_3$, $(0,4,1)\text{-}11$, $(10,11,14)\text{-}4$, and $L=\{K_3(\infty_2,\infty_3,\infty_4)\}$.

4. In 3 replace $(\infty_4,2,\infty_1)\text{-}\infty_3$ and $(\infty_2,\infty_3,\infty_4)$ with $(\infty_1,2,\infty_4)\text{-}\infty_3$ and $L=\{P_4(\infty_1,\infty_3,\infty_2,\infty_4)\}$.

5. In 1 replace $(0,10,\infty_1)\text{-}2$, $(\infty_2,\infty_4,\infty_1)\text{-}\infty_3$, $(2,6,3)\text{-}13$, $\{K_2(2,12),P_3(\infty_2,\infty_3,\infty_4)\}$ with $(0,10,\infty_1)\text{-}\infty_3$, $(\infty_1,\infty_4,\infty_2)\text{-}\infty_3$, $(3,6,2)\text{-}\infty_1$, and $L=\{K_2(2,12),K_2(3,13),K_2(\infty_3,\infty_4)\}$.

In every of the previous cases $T=\{0,1,2,3,11,\infty_1,\infty_2,\infty_3\}$ is a TBS.

$n=20$:

1. $X=Z_{15}\cup\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$; $\mathscr{P}=\{(2+i,9+i,i)\text{-}\infty_1,(1+i,4+i,i)\text{-}\infty_5:i\in Z_{15}\}\cup\{(i,5+i,\infty_2)\text{-}(10+i),(i,10+i,\infty_3)\text{-}(5+i),(5+i,10+i,\infty_4)\text{-}i:0\leqslant i\leqslant 4\}\cup\{(\infty_4,\infty_2,\infty_5)\text{-}\infty_3,(\infty_3,\infty_2,\infty_1)\text{-}\infty_5\}$; $L=\{P_3(\infty_3,\infty_4,\infty_1)\}$.

2. Replace $(\infty_4,\infty_2,\infty_5)\text{-}\infty_3$ and $L=\{P_3(\infty_1,\infty_4,\infty_3)\}$ with $(\infty_5,\infty_2,\infty_4)\text{-}\infty_3$ and $L=\{K_2(\infty_1,\infty_4),K_2(\infty_3,\infty_5)\}$.

In both of the cases $T=\{0,3,6,9,12,\infty_1,\infty_2,\infty_5\}$ is a TBS.

$n=21$:

1. $X=Z_{17}\cup\{\infty_1,\infty_2,\infty_3,\infty_4\}$; $\mathscr{P}=\{(3+i,8+i,i)\text{-}\infty_3:i\in Z_{17}\}\cup\{(4+i,10+i,i)\text{-}\infty_4:0\leqslant i\leqslant 15\}\cup\{(\infty_1,2i,1+2i)\text{-}(3+2i):0\leqslant i\leqslant 7\}\cup\{(\infty_2,1+2i,2+2i)\text{-}(4+2i):0\leqslant i\leqslant 6\}\cup\{(3,9,16)\text{-}0,(15,16,\infty_2)\text{-}\infty_4,(\infty_1,\infty_3,\infty_2)\text{-}0,(16,\infty_1,\infty_4)\text{-}\infty_3\}$; $L=\{K_2(0,2),K_2(1,16)\}$.

2. Replace $(3,9,16)\text{-}0$ and $\{K_2(0,2),K_2(1,16)\}$ with $(3,9,16)\text{-}1$ and $L=\{P_3(2,0,16)\}$.

In both of the cases $T=\{1,2,3,7,\infty_1,\infty_2,\infty_3,\infty_4\}$ is a TBS.

$n=22$:

1. $X=Z_{19}\cup\{\infty_1,\infty_2,\infty_3\}$, $\mathscr{P}=\{(2+i,8+i,i)\text{-}\infty_2,(3+i,10+i,i)\text{-}\infty_3:i\in Z_{19}\}\cup\{(2+i,6+i,1+i)\text{-}\infty_1:0\leqslant i\leqslant 17\}\cup\{(\infty_3,\infty_2,\infty_1)\text{-}0\}$; $L=\{K_3(1,5,0)\}$.

2. In 1 replace $(6,10,5)\text{-}\infty_1$ and $(1,5,0)$ with $(6,10,5)\text{-}1$ and $L=\{P_4(1,0,5,\infty_1)\}$.

3. In 1 replace $(5,11,3)\text{-}\infty_2$ and $(1,5,0)$ with $(3,11,5)\text{-}1$ and $L=\{K_2(3,\infty_2),P_3(1,0,5)\}$.

4. In 1 replace $(\infty_2,\infty_3,\infty_1)\text{-}0$, $(2,8,0)\text{-}\infty_2$, $(3,10,0)\text{-}\infty_3$, $(5,12,2)\text{-}\infty_3$, and $(1,5,0)$ with $(0,\infty_2,\infty_1)\text{-}\infty_3$, $(1,5,0)\text{-}8$, $(3,10,0)\text{-}2$, $(5,12,2)\text{-}8$, and $L=\{K_{1,3}(\infty_3;0,2,\infty_2)\}$.

5. In 1 replace $(6,10,5)\text{-}\infty_1$, $(0,7,16)\text{-}\infty_3$, and $(1,5,0)$ with $(6,10,5)\text{-}1$, $(7,16,0)\text{-}5$, and $L=\{K_2(0,1),K_2(5,\infty_1),K_2(16,\infty_3)\}$.

In every of the previous cases $T=\{1,2,4,7,9,\infty_1,\infty_2,\infty_3\}$ is a TBS.

$n=23$:

$X=Z_{19}\cup\{\infty_1,\infty_2,\infty_3,\infty_4\}$; $\mathscr{P}=\{(3+i,8+i,2+i)\text{-}\infty_1,(4+i,12+i,2+i)\text{-}\infty_2,(5+i,9+i,2+i)\text{-}\infty_3:0\leqslant i\leqslant 16,\ i\neq 4,7,8,12,13\}\cup\{(10+i,2+i,i)\text{-}\infty_4,(7+i,i,3+i)\text{-}\infty_4:i=0,1,9,10\}\cup\{(\infty_1,1+i,i)\text{-}\infty_3,(\infty_3,6+i,1+i)\text{-}\infty_2,(7+i,1+i,2+i)\text{-}\infty_4,(12+i,6+i,7+i)\text{-}\infty_4,(16+i,6+i,8+i)\text{-}\infty_4,(13+i,9+i,6+i)\text{-}\infty_4:i=0,9\}\cup\{(15,1,14)\text{-}\infty_1,(17,2,14)\text{-}\infty_3,(14,16,5)\text{-}\infty_4,(\infty_2,9,15)\text{-}\infty_1,(\infty_1,6,\infty_2)\text{-}0,(14,\infty_4,\infty_2)\text{-}\infty_3,(\infty_1,\infty_3,\infty_4)\text{-}18\}$; $L=\{K_2(0,6)\}$.

$T=\{\infty_1,\infty_2,\infty_3,\infty_4,0,3,4,8\}$ is a TBS. $T'=\{\infty_1,\infty_2,\infty_3,\infty_4,3,4,5,8\}$ is a BS (not TBS). $\square$

Lemma 3.5. *There exists a kite system of* $K_{8+r}\backslash K_r$ $(X, \mathscr{P})$ *with hole H for* $r=0, 1, 2, 3, 4, 5, 6, 7$ *admitting a BS of size* 4 *with no vertices in the hole.*

Proof. When $r=0, 1$, the conclusion follows by Lemma 3.4.

$n=10$:

$X=Z_8\cup\{\infty_1, \infty_2\}$; $\mathscr{P}=\{(1+i, 3+i, i)\text{-}\infty_1 : i\in Z_8\}\cup\{(4, 0, \infty_2)\text{-}3, (5, 1, \infty_2)\text{-}7, (6, 2, \infty_2)\text{-}\infty_1\}$; $H=\{3, 7\}$.

$T=\{\infty_1, \infty_2, 0, 6\}$ is a BS (see example $n=10$ in Lemma 3.4).

$n=11$:

$X=Z_8\cup\{\infty_1, \infty_2, \infty_3\}$, $\mathscr{P}=\{(2+i, 4+i, 1+i)\text{-}\infty_1 : 0\leqslant i\leqslant 6,\ i\neq 4\}\cup\{(4+i, \infty_2, i)\text{-}\infty_3 : 0\leqslant i\leqslant 3\}\cup\{(\infty_1, \infty_2, \infty_3)\text{-}4, (5, 6, \infty_3)\text{-}7, (\infty_1, 5, 0)\text{-}6\}$, $H=\{0, 1, 3\}$.

$T=\{2, \infty_1, \infty_2, \infty_3\}$ is a BS.

$n=12$:

$X=Z_8\cup\{\infty_1, \infty_2, \infty_3, \infty_4\}$; $\mathscr{P}=\{(\infty_1, 2i, 1+2i)\text{-}(3+2i), (\infty_2, 1+2i, 2+2i)\text{-}(4+2i), (\infty_3, 4+i, i)\text{-}(3+i) : 0\leqslant i\leqslant 3\}\cup\{(\infty_4, 4, 7)\text{-}2, (0, 5, \infty_4)\text{-}2, (1, 6, \infty_4)\text{-}3\}$; $H=\{\infty_1, \infty_2, \infty_3, \infty_4\}$.

$T=\{1, 3, 5, 7\}$ is a BS.

$n=13$:

$X=Z_8\cup\{\infty_1, \infty_2, \ldots, \infty_5\}$; $\mathscr{P}=\{(\infty_3, 2i, 1+2i)\text{-}(3+2i), (\infty_4, 1+2i, 2+2i)\text{-}(4+2i) : 0\leqslant i\leqslant 3\}\cup\{(0, 3, \infty_1)\text{-}6, (\infty_1, 4, 1)\text{-}6, (2, 5, \infty_1)\text{-}7, (3, 6, \infty_2)\text{-}1, (\infty_2, 4, 7)\text{-}3, (5, 0, \infty_2)\text{-}2, (0, 4, \infty_5)\text{-}3, (1, 5, \infty_5)\text{-}7, (\infty_5, 6, 2)\text{-}7\}$; $H=\{\infty_1, \infty_2, \ldots, \infty_5\}$.

$T=\{1, 3, 5, 7\}$ is a BS.

$n=14$:

$X=Z_8\cup\{\infty_1, \infty_2, \ldots, \infty_6\}$; $\mathscr{P}=\{(\infty_1, 4+i, i)\text{-}\infty_2 : 0\leqslant i\leqslant 3\}\cup\{(\infty_3, i, 5+i)\text{-}\infty_2, (\infty_4, 2+i, 4+i)\text{-}(6+i), (\infty_5, 5+i, 2+i)\text{-}0 : 0\leqslant i\leqslant 1\}\cup\{(\infty_3, 7, 2)\text{-}1, (\infty_3, 3, 4)\text{-}\infty_2, (\infty_4, 0, 6)\text{-}7, (\infty_4, 7, 1)\text{-}3, (\infty_5, 0, 7)\text{-}\infty_2, (\infty_5, 1, 4)\text{-}7, (0, 1, \infty_6)\text{-}6, (2, 3, \infty_6)\text{-}7, (\infty_6, 4, 5)\text{-}6\}$; $H=\{\infty_1, \infty_2, \ldots, \infty_6\}$.

$T=\{4, 5, 6, 7\}$ is a BS.

$n=15$:

$X=Z_8\cup\{\infty_0, \infty_1, \ldots, \infty_6\}$; $\mathscr{P}=\{(2+i, 4+i, \infty_i)\text{-}(5+i), (3+i, 6+i, \infty_i)\text{-}(7+i) : 0\leqslant i\leqslant 6\}\cup\{(\infty_{3+i}, 3+i, 4+i)\text{-}i : 0\leqslant i\leqslant 3\}\cup\{(\infty_0, 1, 0)\text{-}7, (\infty_1, 2, 1)\text{-}3, (\infty_2, 3, 2)\text{-}5\}$; $H=\{\infty_0, \infty_1, \ldots, \infty_6\}$.

$T=\{1, 3, 5, 7\}$ is a BS. □

Lemma 3.6. $N(n, 1)\neq\emptyset$ *for every* $n\geqslant 4$.

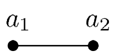
Proof. For $4\leqslant n\leqslant 23$ the conclusion follows from Lemma 3.4. Now let $n=8k+r$, $k\geqslant 3$ and $0\leqslant r\leqslant 7$. The conclusion follows by Lemmas 3.1, 3.4, 3.5, and Construction 3.3. □

3.3. $\lambda\geqslant 2$

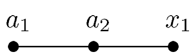
To start with, we note that for $n\equiv 0, 1 \pmod 8$, Lemma 3.6 gives a KS$(n, 1)$ admitting BS. Take λ copies to obtain a KS(n, λ) with BS. For the remaining n, in order to obtain MPK(n, λ) admitting TBS when $\lambda\geqslant 2$, in each of the MPK$(n, 1)$ $(X, \mathscr{P})$ of Lemma 3.6, we label the elements of the TBS T with $x_1, x_2, \ldots, x_h$ and the elements of $X\backslash T$ with $a_1, a_2, \ldots, a_{n-h}$, and we paste solutions for suitable values of λ_i.

Note that, by means of previous constructions, with a similar proof as in that of Lemma 3.6 we can obtain

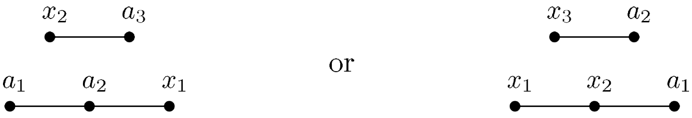
- for every $n=8k+r\geqslant 7$, $r\in\{2, 7\}$, an MPK$(n, 1)$ admitting a BS $T=\{x_1, x_2, \ldots, x_h\}$ with edge-leave:



- for every $n=8k+r\geqslant 12$, $r\in\{4, 5\}$, an MPK$(n, 1)$ admitting a TBS $T=\{x_1, x_2, \ldots, x_h\}$ with edge-leave:

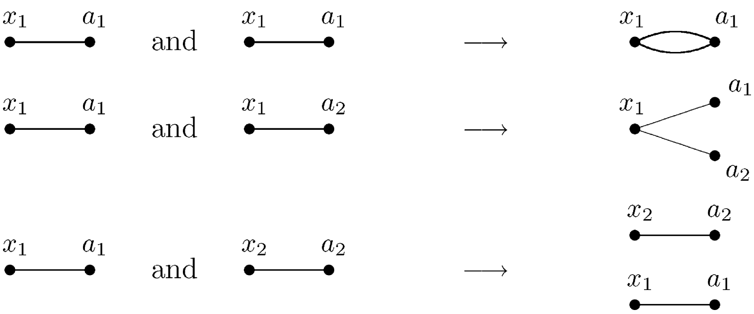


- for every $n = 8k + r \geqslant 6$, $r \in \{3, 6\}$, an MPK$(n, 1)$ admitting a TBS $T = \{x_1, x_2, \ldots, x_h\}$ with edge-leave:

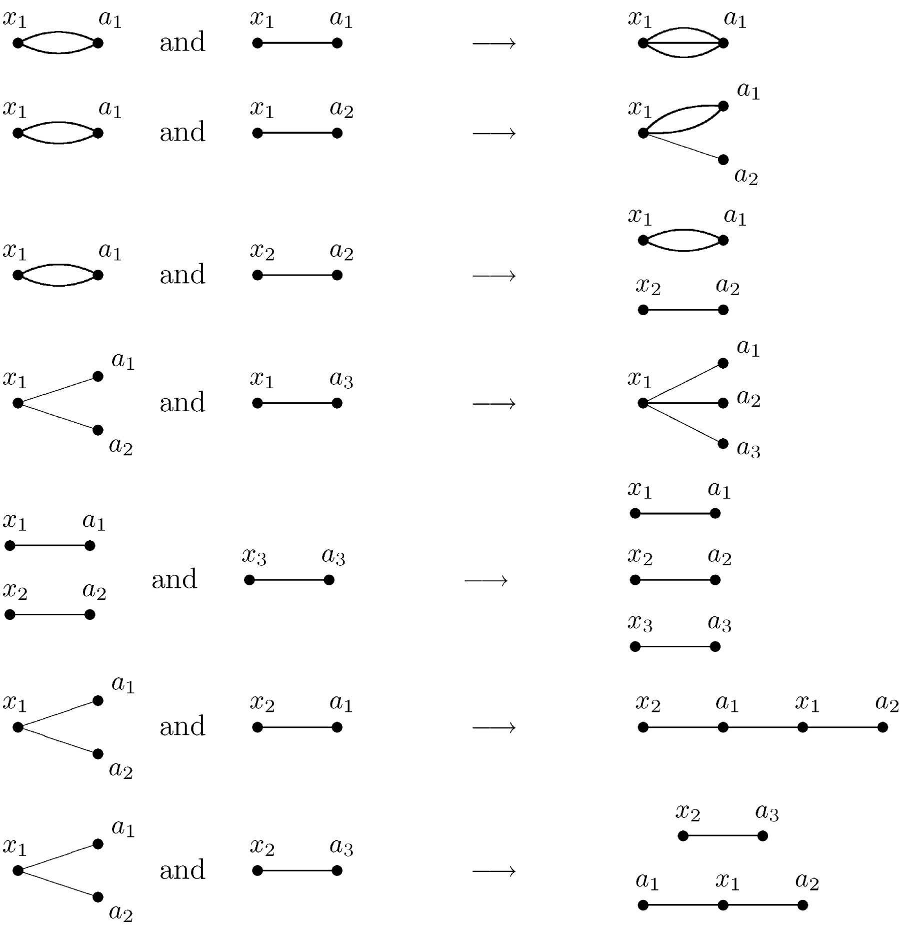


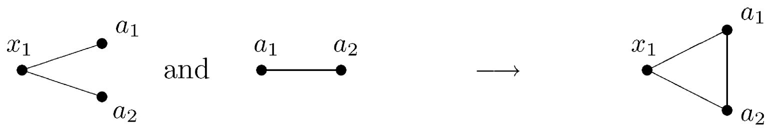
$n = 8k + r \geqslant 7$, $r \in \{2, 7\}$:

$\lambda = 2$: Form MPK$(n, 1)$s with edge-leaves as below and take the union of these to obtain an MPK$(n, 2)$ for each admissible edge-leave:



$\lambda = 3$: Form an MPK$(n, 2)$ and an MPK$(n, 1)$ with edge-leaves as below and take the union of these to obtain an MPK$(n, 3)$ for each admissible edge-leave:



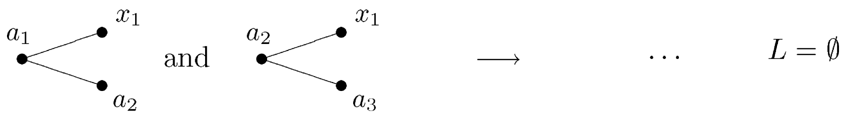


$\lambda = 4$: Form an MPK$(n, 3)$ and an MPK$(n, 1)$ with edge-leaves as below and take the union of these to obtain a KS$(n, 4)$:

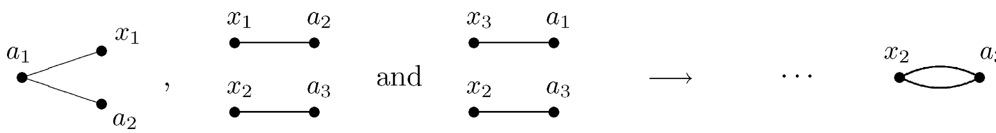
$\lambda = 4k + h$, $0 \leqslant h \leqslant 3$: take the union of k copies of a solution for $\lambda = 4$ and one copy of a solution for $\lambda = h$.

$n = 8k + r \geqslant 12$, $r \in \{4, 5\}$:

$\lambda = 2$: Form MPK$(n, 1)$s with edge-leaves as below and take the union of these to obtain a KS$(n, 2)$:



$\lambda = 3$: Form a KS$(n, 2)$ and an MPK$(n, 1)$ (for each admissible edge-leave) and take the union of these to obtain an MPK$(n, 3)$ for each admissible edge-leave with the exception of the double edge; consider three MPK$(n, 1)$ with the edge-leaves as below and take the union of these to obtain an MPK$(n, 3)$ with edge-leave a double edge:

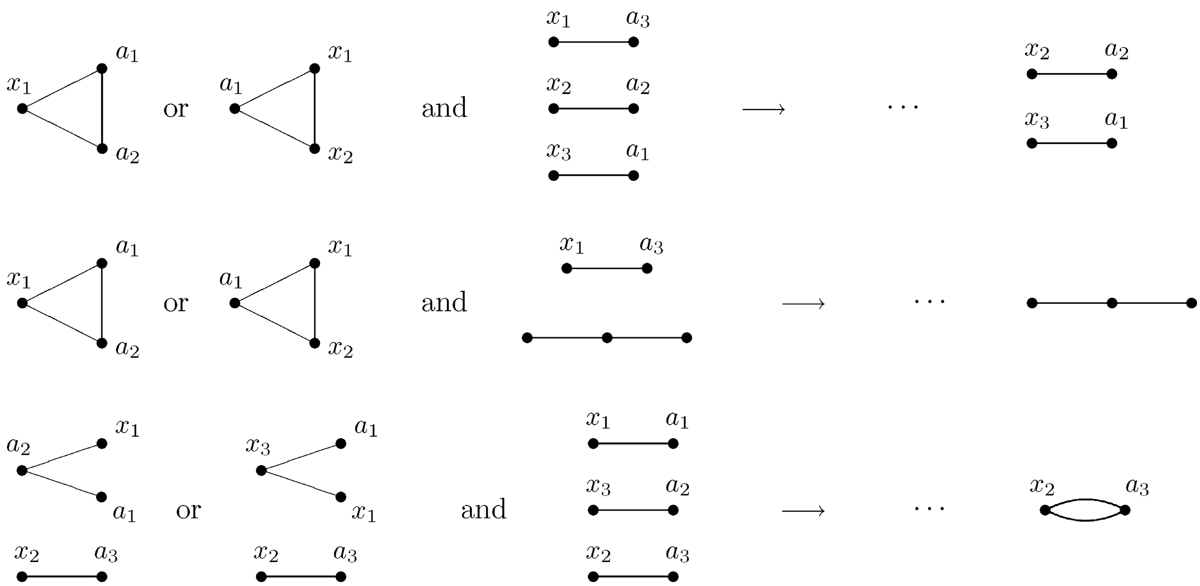


$\lambda = 2k$, $k \geqslant 2$: take the union of k KS$(n, 2)$s to obtain a KS$(n, 2k)$.

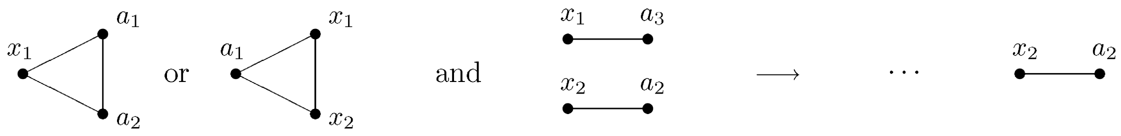
$\lambda = 2k + 1$, $k \geqslant 2$: take the union of $k - 1$ KS$(n, 2)$s and an MPK$(n, 3)$ (for each admissible edge-leave) to obtain an MPK$(n, 2k + 1)$ for each admissible edge-leave.

$n = 8k + r \geqslant 6$, $r \in \{3, 6\}$:

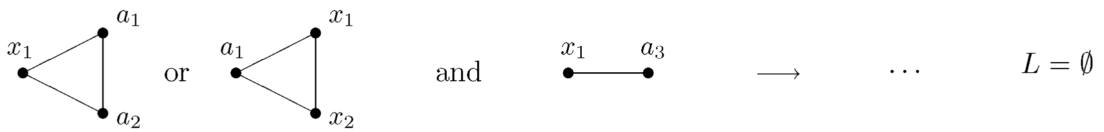
$\lambda = 2$: Form MPK$(n, 1)$s with edge-leaves as below and take the union of these to obtain an MPK$(n, 2)$ for each admissible edge-leave:



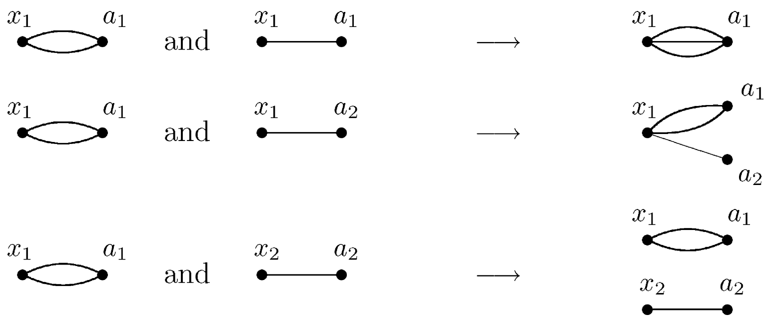
$\lambda = 3$: Form an MPK$(n, 1)$ and an MPK$(n, 2)$ with edge-leaves as below and take the union of these to obtain an MPK$(n, 3)$:



$\lambda = 4$: Form an MPK$(n, 1)$ and an MPK$(n, 3)$ with edge-leaves as below and take the union of these to obtain a KS$(n, 4)$:



$\lambda = 5$: Form a KS$(n, 4)$ and an MPK$(n, 1)$ (for each admissible edge-leave) and take the union of these to obtain an MPK$(n, 5)$ for each admissible edge-leave with no repeated edges; consider an MPK$(n, 2)$ and an MPK$(n, 3)$ with edge-leaves as below and take the union of these to obtain an MPK$(n, 5)$ for each admissible edge-leave containing repeated edges:



$\lambda = 4k + h$, $k \geqslant 1$ and $h = 2, 3, 4, 5$: take the union of k KS$(n, 4)$s and an MPK(n, h) (for each admissible edge-leave) to obtain an MPK$(n, 4k + h)$ for each admissible edge-leave.

Finally, we need consider examples of MPK(n, λ) $(X, \mathscr{P})$ with all admissible edge-leave L for $n = 4$ and 5.

$n = 4$:

$\lambda = 2$: $X = \{0, 1, 2, 3\}$; $\mathscr{P} = \{(2, 3, 1)\text{-}4, (1, 2, 4)\text{-}3, (2, 4, 3)\text{-}1\}$; $L = \emptyset$.

$\lambda = 3$: $X = \{0, 1, 2, 3\}$;

1. $\mathscr{P} = \{(2, 3, 1)\text{-}4, (1, 2, 4)\text{-}3, (2, 4, 3)\text{-}1, (2, 3, 1)\text{-}4\}$; $L = \{P_3(2, 4, 3)\}$;
2. $\mathscr{P} = \{(1, 3, 4)\text{-}2, (3, 4, 2)\text{-}1, (1, 2, 3)\text{-}4, (1, 3, 2)\text{-}4\}$; $L = \{2K_2(1, 4)\}$;
3. $\mathscr{P} = \{(1, 3, 4)\text{-}2, (1, 4, 2)\text{-}3, (2, 4, 3)\text{-}1, (2, 3, 1)\text{-}4\}$; $L = \{K_2(1, 2), K_2(3, 4)\}$.

$\lambda = 2k$ $(k \geqslant 2)$: take k copies of the above MPK$(4, 2)$.

$\lambda = 2k + 1$ $(k \geqslant 2)$: take $k - 1$ copies of the above MPK$(4, 2)$ and one copy for the above MPK$(4, 3)$.

In every case $T = \{1, 3\}$ is a TBS.

$n = 5$:

$\lambda = 2$: $X = \{0, 1, 2, 3, 4\}$; $\mathscr{P} = \{(1, 2, 0)\text{-}3, (2, 3, 4)\text{-}1, (2, 4, 0)\text{-}3, (2, 3, 1)\text{-}0, (3, 1, 4)\text{-}0\}$; $L = \emptyset$.

$\lambda = 3$: $X = \{0, 1, 2, 3, 4\}$;

1. $\mathscr{P} = \{(1, 2, 0)\text{-}3, (2, 3, 4)\text{-}0, (1, 2, 0)\text{-}3, (2, 3, 4)\text{-}1, (2, 4, 0)\text{-}3, (2, 3, 1)\text{-}0, (3, 1, 4)\text{-}0\}$; $L = \{P_3(4, 1, 3)\}$.
2. $\mathscr{P} = \{(1, 2, 0)\text{-}3, (2, 3, 4)\text{-}1, (1, 2, 0)\text{-}3, (2, 3, 4)\text{-}1, (2, 4, 0)\text{-}3, (2, 3, 1)\text{-}0, (3, 1, 4)\text{-}0\}$; $L = \{K_2(4, 0), K_2(1, 3)\}$.
3. $\mathscr{P} = \{(2, 1, 0)\text{-}3, (1, 3, 4)\text{-}0, (2, 3, 0)\text{-}1, (3, 1, 4)\text{-}0, (0, 3, 4)\text{-}1, (1, 3, 2)\text{-}4, (1, 0, 2)\text{-}3\}$; $L = \{2K_2(2, 4)\}$.

$\lambda = 2k$ $(k \geqslant 2)$: take k copies of the above MPK$(5, 2)$.

$\lambda = 2k + 1$ $(k \geqslant 2)$: take $k - 1$ copies of the above MPK$(5, 2)$ and one copy for each MPK$(5, 3)$.

In every case $T = \{1, 4\}$ is a TBS.

Now we are in position to present our main result.

Theorem 3.7. *$N(n, \lambda) \neq \emptyset$ for every $n \geqslant 4$ and $\lambda \geqslant 1$.*

We end this paper with the following research problem: determine the set $N(n, \lambda)$ for each pair of positive integers (n, λ).

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